Self-indexing energy function for Morse-Smale diffeomorphisms on 3-manifolds

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Abstract

The paper is devoted to finding conditions to the existence of a self-indexing energy function for Morse-Smale diffeomorphisms on a 3-manifold M^3. These conditions involve how the stable and unstable manifolds of saddle points are embedded in the ambient manifold. We also show that the existence of a self-indexing energy function is equivalent to the existence of a Heegaard splitting of M^3 of a special type with respect to the considered diffeomorphism.

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Introduction

Let M^n be a smooth closed orientable n-manifold. A diffeomorphism $f : M^n \to M^n$ is called a *Morse-Smale diffeomorphism* if its nonwandering set $\Omega(f)$ consists of finitely many hyperbolic periodic points ($\Omega(f) = Per(f)$) whose invariant manifolds have mutually transversal intersections. D. Pixton [15] defined *a Lyapunov function* for a Morse-Smale diffeomorphism f as a Morse function[1] $\varphi : M^n \to \mathbb{R}$ such that $\varphi(f(x)) < \varphi(x)$ when x is not a peri-

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[1] A function $\varphi : M^n \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate.

odic point and $\varphi(f(x)) = \varphi(x)$ when it is. Such a function can be constructed in different ways[2] (see for instance [12]).

If φ is a Lyapunov function for a Morse-Smale diffeomorphism f, then any periodic point of f is a critical point of φ (see lemma 2.1). The opposite is not true in general since a Lyapunov function may have critical points which are not periodic points of f. Then Pixton [15] defined an *energy function* for a Morse-Smale diffeomorphism f as a Lyapunov function φ such that the critical points of φ coincide with the periodic points of f and proved the following results.

- For any Morse-Smale diffeomorphism given on a surface there is an energy function.
- There is an example of a Morse-Smale diffeomorphism on $\mathbb{S}^3$ which has no energy function.

If p is a periodic point of period k_p for the Morse-Smale diffeomorphism $f : M^n \to M^n$ then the *stable manifold* is $W^s(p) = \{x \in M^n \mid f^{mk_p}(x) \to p$ when $m \to +\infty\}$; the *unstable manifold* is $W^u(p) = \{x \in M^n \mid f^{mk_p}(x) \to p$ when $m \to -\infty\}$. The point p is said to be a *sink* (resp. *source)* when $\dim W^u(p) = 0$ (resp. $\dim W^u(p) = n$). The point p is called a *saddle* point when $\dim W^u(p) \neq 0, n$. A stable (resp. unstable) *separatrix* of the saddle point p is a connected component of $W^s(p) \setminus p$ (resp. $W^u(p) \setminus p$).

Let us recall that a Morse-Smale diffeomorphism $f : M^n \to M^n$ is called *gradient-like* if for any pair of periodic points x, y ($x \neq y$) the condition $W^u(x) \cap W^s(y) \neq \emptyset$ implies $\dim W^s(x) < \dim W^s(y)$. When $n = 3$, a Morse-Smale diffeomorphism is gradient-like if and only if the two-dimensional and one-dimensional invariant manifolds of its different saddle points do not intersect[3].

Let $f : M^n \to M^n$ be a gradient-like diffeomorphism. Then, it follows from [18] (theorem 2.3), that the closure $\bar{\ell}$ of any one-dimensional unstable separatrix ℓ of a saddle point σ is homeomorphic to a segment which consists of this separatrix and two points: σ and some sink ω. Moreover, $\bar{\ell}$ is everywhere smooth except, maybe, at ω. So the topological embedding of $\bar{\ell}$ may be complicated in a neighborhood of the sink.

[2] In 1978 C. Conley [6] proved the existence of a continuous Lyapunov function (that is a function which strictly decreases along orbits outside the chain recurrent set and is constant on components of the chain recurrent set) for any flow (or homeomorphism) given on a compact manifold. This fact was named later the Fundamental Theorem of dynamical systems (see, for example, [16], theorem 1.1, p. 404). Notice, that for Morse-Smale diffeomorphisms the chain recurrent set is exactly the non-wandering set and components of the chain recurrent set are the periodic orbits.

[3] Let us remark that the two-dimensional invariant manifolds of different saddle points of a gradient-like diffeomorphism may have a non empty intersection, namely along the so-called *heteroclinic curves*.

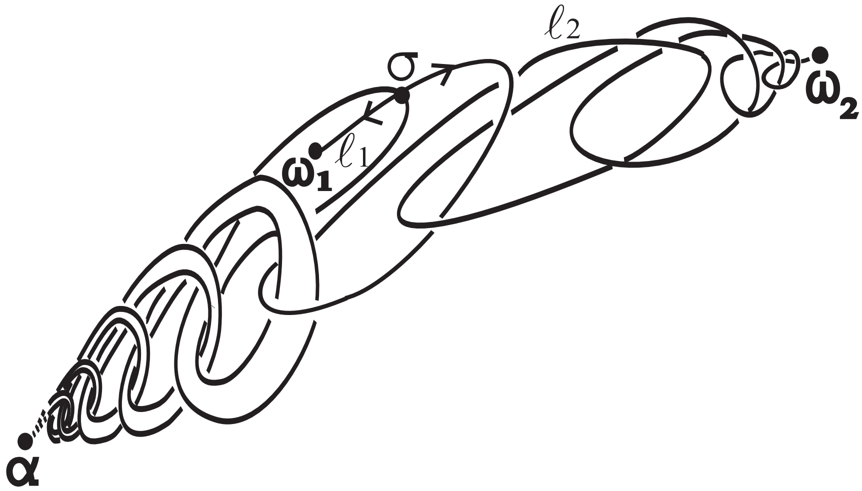

Figure 1: Pixton's example

According to [1], ℓ is called *tame* (or *tamely embedded*) if there is a homeomorphism $\psi : W^s(\omega) \to \mathbb{R}^n$ such that $\psi(\omega) = O$, where O is the origin and $\psi(\bar{\ell} \setminus \sigma)$ is a ray starting from O. In the opposite case ℓ is called *wild*.

In the above mentioned Pixton's example, the non-wandering set of $f : \mathbb{S}^3 \to \mathbb{S}^3$ consists of exactly four fixed points: one source α, two sinks ω_1, ω_2, one saddle σ whose one unstable separatrix ℓ_1 is tamely embedded and the other ℓ_2 is wildly embedded (see fig. 1). Later, the class $\mathcal{G}_4$ of diffeomorphisms on $\mathbb{S}^3$ with such a nonwandering set was considered in [2], where it was proved that, for every diffeomorphism $f \in \mathcal{G}_4$, at least one separatrix ℓ_1 is tame. It was also shown that the topological classification of diffeomorphisms from $\mathcal{G}_4$ is reduced to the embedding classification of the separatrix ℓ_2. Hence it follows that there exist infinitely many diffeomorphisms from $\mathcal{G}_4$ which are not topologically conjugate.

According to Pixton, if the separatrix ℓ_2 is wildly embedded, the Morse-Smale diffeomorphism $f \in \mathcal{G}_4$ has no energy function. The present paper is devoted to finding conditions to the existence of a self-indexing energy function (in the sense of definition 1.2 below) for Morse-Smale diffeomorphisms on 3-manifolds.

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1 Formulation of the results

If φ is a Lyapunov function of a Morse-Smale diffeomorphism $f : M^n \to M^n$ then any periodic point p is a maximum of the restriction of φ to the unstable manifold $W^u(p)$ and a minimum of its restriction to the stable manifold $W^s(p)$ (see lemma 2.1). If these extremums are non-degenerate then the invariant manifolds of p are transversal to all regular level sets of φ in some neighborhood U_p of p. This local property is useful for the construction of a (global) Lyapunov function. So we introduce the following definitions.

Definition 1.1 *A Lyapunov function $\varphi : M^n \to \mathbb{R}$ for a Morse-Smale diffeomorphism $f : M^n \to M^n$ is called a Morse-Lyapunov function if any periodic point p is a non-degenerate maximum of the restriction of φ to the unstable manifold $W^u(p)$ and a non-degenerate minimum of its restriction to the stable manifold $W^s(p)$.*

Theorem 1 *Among the Lyapunov functions of a Morse-Smale diffeomorphism f those which are Morse-Lyapunov form a residual set in the C^∞-topology.*

If p is a critical point of a Morse function $\varphi : M^n \to \mathbb{R}$ then, according to the Morse lemma (see, for example, [13]), in some neighborhood $V(p)$ of p there is a local coordinate system $x_1, \dots, x_n$, named *Morse coordinates*, such that $x_j(p) = 0$ for each $j = \overline{1,n}$ and φ reads $\varphi(x) = \varphi(p) - x_1^2 - \ldots - x_q^2 + x_{q+1}^2 + \ldots + x_n^2$, where q is the index φ at p[4]. It is convenient to deal with *self-indexing* Morse function for which $\varphi(p) = q$.

If φ is a Lyapunov function for a Morse-Smale diffeomorphism $f : M^n \to M^n$ then $q = dim\ W^u(p)$ for any periodic point p of f (see [15], lemma on p. 168). The next definition follows from S. Smale [19] who introduced a similar one for gradient-like vector fields.

Definition 1.2 *A Morse-Lyapunov function φ is called a self-indexing energy function when the following conditions are fulfilled:*

1) the set of the critical points of function φ coincides with the set $Per(f)$ of the periodic points of f;

2) $\varphi(p) = dim\ W^u(p)$ for any periodic point $p \in Per(f)$.

Sometimes we shall speak of a self-indexing energy function even when it is only defined on some domain $N \subset M^n$, meaning that the above conditions hold only for points $x \in N$ such that $f(x) \in N$. In the next results we only deal with 3-dimensional manifolds.

Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism, ω be a sink of f and $L(\omega)$ be the union of all unstable one-dimensional separatrices of saddles which contain ω in their closure. The collection $L(\omega)$ is *tame* if there is a homeomorphism $\varphi : W^s(\omega) \to \mathbb{R}^3$ such that $\varphi(\omega) = O$, where O is the origin

[4]The number of negative eigenvalues of the matrix $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(p)$ is called *the index of the critical point p.*

and $\varphi(\bar{\ell} \setminus \sigma)$ is a ray starting from O for any separatrix $\ell \in L(\omega)$. In the opposite case the set $L(\omega)$ is *wild*. Notice that the tameness of each separatrix $\ell \in L(\omega)$ does not imply the tame property of $L(\omega)$. In [7] there is an example of a wild collection of arcs in $\mathbb{R}^3$ where each arc is tame. Using this example and methods of realization of Morse-Smale diffeomorphisms suggested in [2] and [3], it is possible to construct a gradient-like diffeomorphisms on $\mathbb{S}^3$ having a wild bundle $L(\omega)$.

If $L(\omega)$ consists of exactly one separatrix ℓ then the tame property is equivalent to the existence of a smooth 3-ball $B_\omega \subset W^s(\omega)$ such that $\ell \cap \partial B_\omega$ consists of exactly one point (it follows from a criterion in [9]). Thus we give the following definition.

Definition 1.3 *We say that $L(\omega)$ is almost tamely embedded in M^3 if there is a smooth closed 3-ball $B_\omega \subset W^s(\omega)$ with $\omega \in int\, B_\omega$ such that $\ell \cap \partial B_\omega$ consists of exactly one point for each separatrix $\ell \subset L(\omega)$. If α is a source, there is a similar definition for $L(\alpha)$. We say that the union L of the one-dimensional separatrices is almost tamely embedded in M^3 if $L(\omega)$ and $L(\alpha)$ are so for each sink and source.*

Theorem 2 *If a Morse-Smale diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function then it is gradient-like and the set L of one-dimensional separatrices is almost tamely embedded.*

We would like to understand what conditions could be added to the almost tame embedding property to obtain sufficient conditions for the existence of a self-indexing energy function.

Let $f : M^3 \to M^3$ be a Morse-Smale diffeomorphism. Let us denote by Ω^+ (resp. Ω^-) the set of all sinks (resp. sources), by Σ^+ (resp. Σ^-) the set of all saddle points having one-dimensional unstable (resp. stable) invariant manifolds, by L^+ (resp. L^-) the union of the unstable (resp. stable) one-dimensional separatrices. We set $\mathcal{A}(f) = \Omega^+ \cup L^+ \cup \Sigma^+$, $\mathcal{R}(f) = \Omega^- \cup L^- \cup \Sigma^-$ and $L = L^- \cup L^+$. By construction, $\mathcal{A}(f)$ (resp. $\mathcal{R}(f)$) is a connected set which is an attractor (resp. a repeller)[5] of f. We set

$$g(f) = \frac{|\Sigma^+ \cup \Sigma^-| - |\Omega^+ \cup \Omega^-| + 2}{2},$$

where $|.|$ stands for the cardinality.

We will denote by $\mathcal{H}$ the set of Morse-Smale diffeomorphisms $f : M^3 \to M^3$ with the following properties:

1) f is gradient-like;

[5]A compact set $A \subset M^n$ is *an attractor of a diffeomorphism* $f : M^n \to M^n$ if there is a neighborhood V of the set A such that $f(V) \subset V$ and $A = \bigcap\limits_{n \in \mathbb{N}} f^n(V)$. A set $R \subset M^n$ is called *a repeller* of f if it is an attractor of f^{-1}.

2) the set L of one-dimensional separatrices of f is almost tamely embedded in M^3;

3) $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is diffeomorphic to $S_{g(f)} \times \mathbb{R}$ where $S_{g(f)}$ is an orientable surface of genus $g(f)$[6].

Theorem 3 *If a Morse-Smale diffeomorphism $f : M^3 \to M^3$ belongs to $\mathcal{H}$ then it has a self-indexing energy function.*

It follows from [8] that if the set of one-dimensional separatrices is tamely embedded (that is, $L(\omega)$ and $L(\alpha)$ are tame for each sink ω and source α) then $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is diffeomorphic to $S_{g(f)} \times \mathbb{R}$ and M^3 admits a Heegaard splitting[7] of genus $g(f)$. Thus we get the next result.

Corollary 1.4 *If the set L of one-dimensional separatrices of a gradient-like diffeomorphism $f : M^3 \to M^3$ is tamely embedded, then f has a self-indexing energy function.*

The next theorem gives necessary and sufficient conditions to the existence of a self-indexing energy function by means of special Heegaard splittings of M^3. We also need the following definition.

Definition 1.5 *Let D be a subset of M^n. It is said to be f-compressed when $f(D)$ is contained in the interior of D.*

Theorem 4 *A gradient-like diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function if and only if M^3 is the union of three domains with mutually disjoint interiors, $M^3 = P^+ \cup N \cup P^-$, satisfying the following conditions.*

1) P^+ (resp. P^-) is a f-compressed (resp. f^{-1}-compressed) handlebody of genus $g(f)$ and $\mathcal{A}(f) \subset P^+$ (resp. $\mathcal{R}(f) \subset P^-$);

2) $W^s(\sigma^+) \cap P^+$ (resp. $W^u(\sigma^-) \cap P^-$) consists of exactly one two-dimensional closed disk for each saddle point $\sigma^+ \in \Sigma^+$ (resp. $\sigma^- \in \Sigma^-$);

3) there is a diffeomorphism $q : S_{g(f)} \times [0,1] \to N$ such that $q(S_{g(f)} \times \{t\})$, $t \in [0,1]$ bounds an f-compressed handlebody.

Remark 1.6 Observe that condition 2) implies that the 1-dimensional separatrices are almost tamely embedded. Indeed, if thin neighborhoods of the disks $P^+ \cap W^s(\sigma^+)$, $\sigma^+ \in \Sigma^+$, are removed from P^+, one gets a union of balls whose boundaries fulfill definition 1.3.

[6] Notice that items 1) and 2) do not imply item 3). In section 5 there is an example of a gradient-like diffeomorphism on $M^3 = \mathbb{S}^2 \times \mathbb{S}^1$ whose set of one-dimensional separatrices is almost tamely embedded and such that $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is not a product.

[7] Let us recall that a three-dimensional orientable manifold is called *a handlebody of a genus $g \geq 0$* if it is obtained from a 3-ball by an orientation reversing identification of g pairs of pairwise disjoint 2-discs in its boundary. The boundary of such a handlebody is an orientable surface of genus g. A *Heegaard splitting* of genus $g \geq 0$ for a manifold M^3 is a representation of M^3 as the gluing of two handlebodies of genus g by means of some diffeomorphism of their boundaries. Their common boundary after gluing, a surface of genus g in M^3, is called a *Heegaard surface*.

2 Properties of Lyapunov functions for a Morse-Smale diffeomorphism

Lemma 2.1 *Let $\varphi : M^n \to \mathbb{R}$ be a Lyapunov function for a Morse-Smale diffeomorphism $f : M^n \to M^n$. Then*

1) $-\varphi$ is Lyapunov function for f^{-1};

2) if p is a periodic point of f then $\varphi(x) < \varphi(p)$ for every $x \in W^u(p) \setminus p$ and $\varphi(x) > \varphi(p)$ for every $x \in W^s(p) \setminus p$;

3) if p is a periodic point of f then p is a critical point of φ.

Proof:

1) It follows from the definition that $\varphi(x) \leq \varphi(f^{-1}(x))$ for any x and the equality only holds for the periodic points. By multiplying this inequality by -1 we get the wanted inequality.

2) As the unstable manifold of a periodic point p for f coincides with the stable manifold of p for f^{-1} it is enough to prove items 2) and 3) only for $W^u(p)$. Let $x \in W^u(p) \setminus p$. It follows from the definition of the unstable manifold of a periodic point that $\lim\limits_{m\to\infty} f^{-mk_p}(x) = p$; hence $\lim\limits_{m\to\infty} \varphi(f^{-mk_p}(x)) = \varphi(p)$. We have $\varphi(x) < \varphi(f^{-k_p}(x)) < \ldots < \varphi(f^{-mk_p}(x)) < \ldots$ and, hence, $\varphi(x) < \varphi(p)$.

3) Let us assume that p is a regular point of φ. Thus the level set $\varphi^{-1}(\varphi(p))$ is $(n-1)$-manifold. It follows from point 2) that $T_pW^u(p)$ and $T_pW^s(p)$ must be tangent to $\varphi^{-1}(\varphi(p))$. This is impossible because $T_pW^u(p)$ and $T_pW^s(p)$ generate T_pM^n. $\diamond$

Denote $Ox_1 \ldots x_q = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{q+1} = \ldots = x_n = 0\}$ and $Ox_{q+1} \ldots x_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \ldots = x_q = 0\}$.

Lemma 2.2 *Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism and $\mathcal{O}(p)$ be a periodic orbit of period k_p and $\dim W^u(p) = q$. Then there exist a Morse-Lyapunov function $\varphi_{\mathcal{O}(p)}$ defined on some neighborhood $U_{\mathcal{O}(p)}$ of $\mathcal{O}(p)$ and, for each $r \in \mathcal{O}(p)$, Morse coordinates $x_1, \ldots, x_n$ for $\varphi_{\mathcal{O}(p)}$ near r such that:*

1) $\varphi_{\mathcal{O}(p)}(\mathcal{O}(p)) = q$;

2) $(W^u(r) \cap U_{\mathcal{O}(p)}) \subset Ox_1 \ldots x_q$ and $(W^s(r) \cap U_{\mathcal{O}(p)}) \subset Ox_{q+1} \ldots x_n$.

Proof: As $\mathcal{O}(p)$ is a hyperbolic set then, for each $r \in \mathcal{O}(p)$, there is a splitting of the tangent space T_rM^n as a direct sum $T_rM^n = T_rW^u(r) \oplus T_rW^s(r)$ such that $Df_r(T_rW^u(r)) = T_{f(r)}W^u(f(r))$ and $Df_r(T_rW^s(r)) = T_{f(r)}W^s(f(r))$. Moreover, there is a metric $\|\cdot\|$ on M^n such that for some λ, $0 < \lambda < 1$, we have:

$$\left\|Df^{-1}(v^u)\right\| \leq \lambda \left\|v^u\right\|,$$

$$\left\|Df(v^s)\right\| \leq \lambda \left\|v^s\right\|,$$

for each $v^u \in E^u$ and $v^s \in E^s$, where $E^u = \bigcup_{r\in\mathcal{O}(p)} T_rW^u(r)$ and $E^s = \bigcup_{r\in\mathcal{O}(p)} T_rW^s(r)$[8].

Let us define $\varphi : E^u \oplus E^s \to \mathbb{R}$ by the formula

$$\varphi(v^u, v^s) = q - \|v^u\|^2 + \|v^s\|^2 .$$

Let us check that $\varphi(Df(v^u, v^s)) < \varphi(v^u, v^s)$ for all non-zero $v^u \in E^u$ or $v^s \in E^s$. Indeed, $\varphi(Df(v^u, v^s)) - \varphi(v^u, v^s) = -\|Df(v^u)\|^2 + \|Df(v^s)\|^2 + \|v^u\|^2 - \|v^s\|^2 \leq -\frac{1}{\lambda^2}\|v^u\|^2 + \lambda^2\|v^s\|^2 + \|v^u\|^2 - \|v^s\|^2 \leq -(\frac{1}{\lambda^2} - 1)\|v^u\|^2 - (1-\lambda^2)\|v^s\|^2 < 0$ for all non-zero $v^u \in E^u$ and $v^s \in E^s$.

Identify a small neighborhood $U_{\mathcal{O}(p)}$ of $\mathcal{O}(p)$ with a neighborhood of the zero-section of $E^u \oplus E^s$ by a diffeomorphism which maps the local unstable (resp. stable) manifold into E^u (resp. E^s). For every $v = (v^u, v^s) \in U_{\mathcal{O}(p)}$ we have $f(v^u, v^s) = Df(v^u, v^s) + o(v)$. Therefore $\varphi(f(v^u, v^s)) < \varphi(v^u, v^s)$ for all non-zero $(v^u, v^s) \in U_{\mathcal{O}(p)}$ if this neighborhood is chosen small enough. Hence φ is the desired function. ◇

¿From lemma 2.2 we deduce the following genericity theorem.

Theorem 1 *Among the Lyapunov functions of a Morse-Smale diffeomorphism f those which are Morse-Lyapunov form a residual set in the C^∞-topology.*

Proof: We recall that a property is said *generic* when it is shared by all points in a *residual subset* (*i.e.* a set which is a countable intersection of dense open subsets). Let us show that in the set of Lyapunov functions for f, the set of Morse-Lyapunov functions for f is open and dense; hence, it is residual. Here, our property is clearly open. Let us show it is dense. Let us consider φ, a Lyapunov function for the Morse-Smale diffeomorphism f. In some open neighborhood U of $Per(f)$ take any function $\varphi_{_{Per(f)}}$ which is a Morse-Lyapunov function of f; it exists according to lemma 2.2. Let $\widetilde{U} \subset int\ U$ be a closed neighborhood of $Per(f)$, $c > 0$ and $v(x) : M^n \to \mathbb{R}$ be a C^∞-function such that $0 \leq v(x) \leq c$, $v(x) \equiv c$ on $\widetilde{U}$ and $v(x) \equiv 0$ out of U. One checks that $\varphi_c := \varphi + v.\varphi_{_{Per(f)}}$ is a Morse-Lyapunov function when c is a small enough positive constant. Indeed, if p is a periodic point, $\varphi|_{W^s(p)}$ has a minimum at p (lemma 2.1) and $\varphi_{_{Per(f)}}|_{W^s(p)}$ has a non-degenerate minimum. Hence $\varphi_c|_{W^s(p)}$ has a non-degenerate minimum at p for any $c > 0$[9]. Similarly, $\varphi_c|_{W^u(p)}$ has a

[8] Such a metric is called *Lyapunov metric*, see, for example, [11]).

[9] As p is a critical point for functions $\varphi|_{W^s(p)}(x)$ and $\varphi_{_{Per(f)}}|_{W^s(p)}(x)$ then they have forms $\varphi|_{W^s(p)}(x) = \varphi|_{W^s(p)}(p) + Q_1(x) + P_1(x)$ and $\varphi_{_{Per(f)}}|_{W^s(p)}(x) = \varphi_{_{Per(f)}}|_{W^s(p)}(p) + Q_2(x) + P_2(x)$, where $Q_1(x), Q_2(x)$ are quadratic forms and $P_1(x), P_2(x)$ satisfy to condition $\lim\limits_{\|x\|\to 0} \frac{P_i(x)}{\|x\|^2} = 0$, $i = 1, 2$. As p is a minimum for $\varphi|_{W^s(p)}(x)$ then $Q_1(x) \geq 0$ for $x \neq p$, as p is a non-degenerate minimum for $\varphi_{_{Per(f)}}|_{W^s(p)}(x)$ then $Q_2(x) > 0$ for $x \neq p$ and, hence,

non-degenerate maximum at p. As T_pM^n is the direct sum of $T_pW^u(p)$ and $T_pW^s(p)$, then φ_c is non-degenerate at p. Each point in $Per(f)$ is a non-degenerate critical point of φ_c, for any $c > 0$. Moreover, since the sum of two Lyapunov functions is a Lyapunov function, there is some open neighborhood $\widehat{U}$ of $Per(f)$ on which φ_c is a Morse-Lyapunov function for any $c > 0$.

Besides, there is a positive ε such that, for every $x \notin \widehat{U} \cap f^{-1}(\widehat{U})$, one has $\varphi(x) > \varphi(f(x)) + \varepsilon$. Thus, whatever $\varphi_{_{Per(f)}}$ is, there exists a small c so that φ_c fulfills the Lyapunov inequality for every $x \notin \widehat{U} \cap f^{-1}(\widehat{U})$. If φ_c is a Morse function then the proof is finished. If φ_c is not a Morse function then a last C^∞-approximation of φ_c, relatively to $\widehat{U}$, makes it a Morse-Lyapunov function everywhere.

$\diamond$

3 Necessary conditions to the existence of a self-indexing energy function

Theorem 2 *If a Morse-Smale diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function then it is gradient-like and the set L of one-dimensional separatrices is almost tamely embedded.*

To prove this theorem we need the two next lemmas.

Lemma 3.1 *If a Morse-Smale diffeomorphism $f : M^n \to M^n$ has a self-indexing energy function φ then it is gradient-like.*

Proof: Assume the contrary: a Morse-Smale diffeomorphism $f : M^n \to M^n$ has a self-indexing energy function $\varphi : M^n \to \mathbb{R}$ and it is not gradient-like. Then there are points $x, y \in Per(f)$ ($x \neq y$) such that $W^u(x) \cap W^s(y) \neq \emptyset$ and $\dim W^s(x) \geq \dim W^s(y)$. Put $\dim W^u(x) = k$, $\dim W^u(y) = m$ and $z \in W^u(x) \cap W^s(y)$. As $n - k = \dim W^s(x) \geq \dim W^s(y) = n - m$ then $k \leq m$. According to lemma 2.1, $\varphi(z) < \varphi(x) = \dim W^u(x) = k$, $\varphi(z) > \varphi(y) = \dim W^u(y) = m$, hence, $k > m$. This is a contradiction. $\diamond$

Lemma 3.2 *If a Morse-Smale diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function φ then the union of its one-dimensional separatrices are almost tamely embedded in M^3.*

$Q_1(x) + c \cdot Q_2(x) > 0$ for $x \neq p$. It follows from reducibility of positive-definite quadratic form to sum of squares of all coordinates that $Q_1(x) + c \cdot Q_2(x)$ is non-degenerate. Thus $\varphi_c|_{W^s(p)}$ has non-degenerate minimum at the point p.

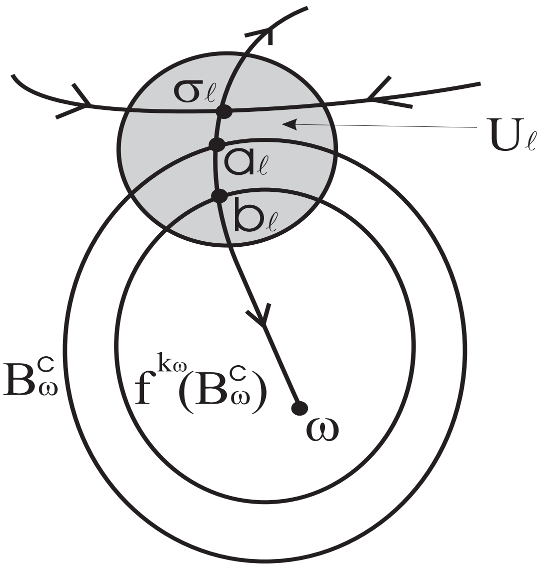

Figure 2: One-dimensional separatrix and self-indexing energy function

Proof: We shall only give a proof[10] for a sink. Let ω be a sink of period k_ω, $L(\omega)$ be the union of all unstable one-dimensional separatrices whose closure contains ω. According to lemma 3.1, f is gradient-like and, hence, for any separatrix $\ell \in L(\omega)$ its closure $\bar{\ell}$ consists of $\ell \cup \{\omega, \sigma_\ell\}$, where σ_ℓ is a saddle point of f. As φ is a Morse-Lyapunov function then in some neighborhood U_ℓ of σ_ℓ equipped with Morse coordinates we have $\varphi(x) = 1 - x_1^2 + x_2^2 + x_3^2$ and $W^u(\sigma_\ell)$ is transversal to the regular level sets of φ in U_ℓ. Let U be the union of the U_ℓ's for $\ell \subset L(\omega)$.

Let k be an integer so that f^k leaves each ℓ invariant. Since the action of f^k on ℓ is discrete it has a fundamental domain $[a_\ell, b_\ell] \subset U$, hence transversal to the level sets of φ. More precisely, $\varphi|_{[a_\ell,b_\ell]}$ has no critical points and $1 = \varphi(\sigma_\ell) > \varphi(a_\ell) > \varphi(b_\ell)$. We may choose all the a_ℓ's in the same level set of φ. We will show that the level set of $\varphi|_{W^s(\omega)}$ containing the a_ℓ's is a 2-sphere crossing ℓ at a_ℓ only.

As φ is a self-indexing energy function for f then ω is the unique critical point of $\varphi|_{W^s(\omega)}$ and its index is 0. Moreover, in some neighborhood $V(\omega)$ of ω, equipped with Morse coordinates, φ reads $\varphi(x) = x_1^2 + x_2^2 + x_3^2$ and hence every regular level set of φ near ω is a smooth 2-sphere which bounds a smooth 3-ball containing ω. According to the Morse theory[11], for any value $c \in (0, 1)$, $\varphi^{-1}(c) \cap W^s(\omega)$ is also a smooth 2-sphere S^c_ω which bounds a smooth 3-ball

[10] It is mainly the proof of proposition 2 in Pixton's article [15], except that our conclusion is slightly stronger than his; we also take advantage of the fact that our energy function is generic.

[11] If $\varphi^{-1}[a, b]$ is compact and does not contain critical points, then $\varphi^{-1}(-\infty, a]$ is diffeomorphic to $\varphi^{-1}(-\infty, b]$ (see, for example, Theorem 3.1 in [13]).

$B^c_\omega \subset W^s(\omega)$ such that $\omega \in f^k(B^c_\omega) \subset int\ B^c_\omega$. We choose $c = \varphi^{-1}(a_\ell)$, a value which does not depend on ℓ (see figure 2).

Assume that there is one point $y \neq a_\ell$ in $\ell \cap S^c_\omega$; certainly y belongs to the interval (a_ℓ, ω) in ℓ. We have $y = f^{mk}(x)$ for some $x \in [a_\ell, b_\ell]$ and some positive integer m. By the Lyapunov property we have $c = \varphi(y) < \varphi(x) < \varphi(a_\ell) = c$, which is a contradiction. $\diamond$

Thus, theorem 2 follows directly from lemmas 3.1 and 3.2. The next lemma is useful for the proof of the necessary conditions in theorem 4 (see lemma 3.4).

Lemma 3.3 *For any Morse-Smale diffeomorphism $f : M^3 \to M^3$*

$$|\Sigma^-| - |\Omega^-| + 1 = |\Sigma^+| - |\Omega^+| + 1 = g(f).$$

Proof: According to [17], a Morse-Smale diffeomorphism induces in all homology groups isomorphisms whose eigenvalues are roots of unity. Thus there is an integer k such that f^k leaves $Per(f^k)$ fixed, $f^k|_{W^u(p)}$ preserves the orientation of $W^u(p)$ for any point $p \in Per(f^k)$ and 1 is the only eigenvalue of the isomorphism induced by f^k on homology. Applying the Lefschetz formula to f^k yields

$$\sum_{p \in Per(f^k)} (-1)^{dim\ W^u(p)} = \sum_{i=0}^{3} (-1)^i t_i,$$

where t_i is the trace of the map induced by f^k on the i-th homology group $H_i(M, \mathbb{R})$. By assumption on k, t_i coincides with the i-th Betti number and the alternating sum of the t_i's is the Euler characteristic, which is 0 since M is an odd-dimensional, closed oriented manifold. So we get $|\Sigma^-| - |\Omega^-| = |\Sigma^+| - |\Omega^+|$. $\diamond$

Lemma 3.4 *If a Morse-Smale diffeomorphism $f : M^3 \to M^3$ has a self-indexing energy function φ then M^3 is the union of three domains with mututally disjoint interiors, $M^3 = P^+ \cup N \cup P^-$, satisfying the following conditions.*

1) P^+ (resp. P^-) is a f-compressed (resp. f^{-1}-compressed) handlebody of genus $g(f)$ and $\mathcal{A}(f) \subset P^+$ (resp. $\mathcal{R}(f) \subset P^-$);

2) $W^s(\sigma^+) \cap P^+$ (resp. $W^u(\sigma^-) \cap P^-$) consists of exactly one two-dimensional closed disk for each saddle point $\sigma^+ \in \Sigma^+$ (resp. $\sigma^- \in \Sigma^-$);

3) there is a diffeomorphism $q : S_{g(f)} \times [0,1] \to N$ such that $q(S_{g(f)} \times \{t\})$, $t \in [0,1]$ bounds an f-compressed handlebody.

Proof: 1) For $0 < \varepsilon < 1$, we set $P^+_\varepsilon = \varphi^{-1}([0, 1+\varepsilon])$. According to the Morse theory, it is obtained by gluing $|\Sigma^+|$ 1-handles[12] to an union of $|\Omega^+|$

[12] A 3-dimensional 1-handle is the product of an interval with a 2-disc. The gluing is made along the top and bottom disks (see Section 3 in [13]).

3-balls. Moreover, it is connected since M itself is connected and any generic path in M, whose end points are in P^+_ε, may be pushed by the gradient flow of φ into P^+_ε. Therefore P^+_ε is a handlebody of genus $|\Sigma^+| - |\Omega^+| + 1$, that is $g(f)$ according to lemma 3.3. As φ is a Lyapunov function, P^+_ε is f-compressed. By definition of a self-indexing energy function and lemma 2.1, $\varphi(\mathcal{A}(f)) = [0,1]$ and, consequently $\mathcal{A}(f) \subset int\ P^+_\varepsilon$. Similarly, using the diffeomorphism f^{-1}, we get that $P^-_\varepsilon = \varphi^{-1}([0, 2-\varepsilon])$ is a handlebody of genus $g(f)$ which f^{-1}-compressed and contains $\mathcal{R}(f)$ in its interior.

2) As φ is a Morse-Lyapunov function then, for a small enough $\varepsilon_0 \in (0, \frac{1}{2})$, the handlebodies $P^+ = P^+_{\varepsilon_0}$ and $P^- = P^-_{\varepsilon_0}$ satisfy the following: $P^+ \cap W^s(\sigma^+)$ (resp. $P^- \cap W^u(\sigma^-)$) consists of exactly one two-dimensional disk D_{σ^+} (resp. D_{σ^-}) for any $\sigma^+ \in \Sigma^+$ (resp. $\sigma^- \in \Sigma^-$).

3) We take $N = \varphi^{-1}([1+\varepsilon_0, 2-\varepsilon_0])$. As φ has no critical points on N and is constant on each boundary component, N satisfies to condition 3). $\diamond$

4 Construction of a self-indexing energy function for a gradient-like diffeomorphism in dimension 3

In this section, the considered manifold is 3-dimensional and $f : M^3 \to M^3$ is a gradient-like Morse-Smale diffeomorphism whose set L of all 1-dimensional separatrices are almost tamely embedded.

4.1 Auxiliary lemmas

Lemma 4.1 *Let ω be a sink of period k_ω of f and $L(\omega)$ be the set of the 1-dimensional separatrices ending at ω. Then there exists a smooth closed 3-ball $B \subset W^s(\omega)$, $\omega \in int\, B$, such that:*

1) B is f^{k_ω}-compressed;

2) for any $\ell \subset L(\omega)$ the sphere $S = \partial B$ crosses ℓ at one point a_ℓ only and transversely.

Proof: For simplicity we make $k_\omega = 1$. By definition, there exists a closed ball $B_0 \subset W^s(\omega)$ whose boundary S_0 meets condition 2). If S is an embedded sphere in $W^s(\omega)$ then $B(S)$ will denote the ball it bounds. If S meets condition 2), then $\omega \in int\, B(S)$.

Let m be the first integer such that $f^k(S_0) \cap S_0 = \emptyset$ for all $k > m$. For any $x \in S_0$ we choose a compact neighborhood K_x of x in $W^s(\omega)$ such that $f(K_x) \cap K_x = \emptyset$; it exists since S_0 avoids the fixed point. From the family of the

K_x's we extract a finite covering $K_1, \ldots, K_p$ of S_0. By a usual transversality theorem ([10], chap. 3), we may approximate S_0 in the C^∞ topology by a sphere having the following property: $f(S_0 \cap K_1)$ is transversal to S_0. A next approximation allows one to get such a transversality along $K_1 \cup K_2$, and so on. Condition 2) is kept when approximating. Thus, in what follows, we may assume that S_0 itself is transversal to its successive images $f(S_0), \ldots, f^m(S_0)$. In the next step, we are going to modify S_0 into S_1 which still fulfills condition 2) and such that $f^k(S_1) \cap S_1 = \emptyset$ for all $k \geq m$. Iterating this process will yield the wanted sphere S. Indeed, as $f(S)$ is disjoint from S and ω is an attractor, we must have $f(S) \subset int\, B(S)$, which means that $B(S)$ is f-compressed.

Assume first $m = 1$ (that is, $f^k(S_0) \subset int\, B_0$ for all $k \geq 2$) and denote $\Sigma = f(S_0)$. Each intersection curve γ in $S_0 \cap \Sigma$ bounds a disk $D \subset \Sigma$. We choose γ to be *innermost* in the sense that the interior of D contains no intersection curves. Then the curve γ bounds a singular disk $D' \subset S_0$ such that $D \cup D'$ is an embedded 2-sphere homotopic to zero in $W^s(\omega) \setminus \{\omega\}$. We notice that D and D' have the same number (0 or 1) of intersection points with any one-dimensional separatrix ℓ, since Σ, as S_0 does, also satisfies condition 2). We define S_0' as the sphere obtained from S_0 by removing the interior of D', gluing D along γ, pushing so that S_0' avoids $D \cup D'$, and smoothing (S_0' still meets condition 2)). Notice that the intersection curves of $f^{-1}(S_0) \cap S_0$ are in bijection by f with the intersection curves of $f(S_0) \cap S_0$. It will be useful to perform the above construction with an innermost disk $D \subset f^{-1}(S_0)$ instead of $D \subset f(S_0)$. In both cases, we have to check:

(i) $f^k(S_0') \subset B(S_0')$ for all $k \geq 2$ (which is equivalent to $f^{-k}(S_0') \cap S_0' = \emptyset$);

(ii) there are less intersection curves in $f(S_0') \cap S_0'$ than in $f(S_0) \cap S_0$.

Point (ii) is not always true; it depends on the position of D with respect to B_0. But we shall prove that there always exists an innermost disk D, in $f(S_0)$ or in $f^{-1}(S_0)$, such that (ii) is satisfied.

Case 1: $D \subset f(S_0)$ and $D \cap int\, B_0 = \emptyset$. Forgetting the pushing-smoothing, for $k > 1$ we have $f^k(S_0') \subset f^k(S_0) \cup f^k(\Sigma) = f^k(S_0) \cup f^{k+1}(S_0) \subset int\, B_0 \subset B(S_0')$, hence (i) holds. We also have $f(S_0') \cap S_0' \subset (f(S_0) \cup f^2(S_0)) \cap S_0' \subset f(S_0) \cap S_0'$, as $f^2(S_0)$ lies in the interior of B_0 and D in its exterior. Hence, (ii) holds after pushing-smoothing.

Case 1': $D \subset f^{-1}(S_0) \cap B_0$. The proof of (i) and (ii) in this case is similar to the previous one in replacing the positive iterates of f by the negative iterates.

Case 2: $D \subset f(S_0) \cap B_0$ and $D \cap f^2(B_0) = \emptyset$. We have $f^{2k}(S_0') \subset B(f^2(S_0))$, hence disjoint from S_0' for all $k > 0$. Similarly, $f^3(S_0')$ lies in $f^3(B_0) \subset int\,(f(B_0) \cap B_0)$, thus it does not intersect D and (i) holds. Before pushing-smoothing, we have $f(S_0') \subset f(S_0) \cup f(D)$. As $f(D)$ lies in $f^2(B_0)$, we have $f(S'_0) \cap S'_0 \subset f(S_0) \cap S'_0$. Hence, pushing decreases the number of intersection curves and (ii) holds.

Case 2': $D \subset f^{-1}(S_0) \cap int\, f^{-2}(B_0)$ and $D \cap int\, B_0 = \emptyset$. By using the negative iterates of f one proves that points (i) and (ii) hold.

Case 3: $D \subset f(S_0) \cap B_0$ and $D \cap f^2(B_0) \neq \emptyset$. We look at the intersection curves of D with $f^2(S_0)$ and choose one of them, α, which is innermost on D: $\alpha = \partial d$ with $d \subset D$. There is a unique disk d' on $f^2(S_0)$ such that the embedded sphere $d \cup d'$ does not surround ω. There are two subcases: (a) $d \subset f^2(B_0)$ and (b) $d \subset int\, B_0 \setminus int\, f^2(B_0)$. When (a), $f^{-2}(d)$ meets the condition of case 1' and, when (b), it meets the condition of case 2'. In both subcases, points (i) and (ii) hold for this innermost disk. Finally, in any case it is possible to reduce the number of intersection curves of S_0 with its image, keeping condition 2).

Repeating this process yields S_1, a sphere meeting condition 2) and such that $f(S_1) \cap S_1 = \emptyset$, (which implies that $f(B(S_1))$ is f-compressed).

When $m > 1$, the end of the proof goes as follows. We introduce $g_r = f^{2^r}$. For r big enough, we have $g_r^k(S_0) \cap S_0 = \emptyset$ for all $k \geq 2$. According to what we just explained, after changing S_0 into another sphere S_1 we get $g_r(S_1) \cap S_1 = \emptyset$. This amounts to decrease r by 1: $g_{r-1}^k(S_1) \cap S_1 = \emptyset$ for all $k \geq 2$. Recursively, we find a ball satisfying both required conditions. $\diamond$

We now consider the orbit $\mathcal{O}(\omega)$. We just found a ball $B \subset W^s(\omega)$ such that B lies in the interior of $f^{-k_\omega}(B)$. We choose a sequence $B = B_0 \subset B_1 \subset \ldots \subset B_{k_\omega - 1} \subset f^{-k_\omega}(B_\omega)$ with mutually disjoint boundaries. Set $B_{\mathcal{O}(\omega)} = \bigcup_{j=0}^{k_\omega - 1} f^j(B_j)$. It is clearly f-compressed.

Lemma 4.2 *For each $j = \overline{0, k_\omega - 1}$, let $B_j \subset W^s(f_j(\omega))$ be a ball centered at $f_j(\omega)$. The union $B = \bigcup_{j=0}^{k_\omega - 1} f^j(B_j)$ is assumed to be f-compressed. Then there is a self-indexing energy function $\varphi : B \to \mathbb{R}$ for f having ∂B as a level set.*

Proof: According to lemma 2.2, there is an open neighborhood U of $\mathcal{O}(\omega)$, $U \subset B$, and a self-indexing energy function $\varphi_{\mathcal{O}(\omega)} : U \to \mathbb{R}$ for f. A level set of $\varphi_{\mathcal{O}(\omega)}$ whose value is positive and small is the union of k_ω copies of 2-spheres. For each $j = \overline{0, k_\omega - 1}$, we choose a smooth 3-ball Q_j in U, centered at $f^j(\omega)$, with boundary G_j such that $G = \bigcup_j G_j$ is a level set of $\varphi_{\mathcal{O}(\omega)}$. We denote $Q = \bigcup_j Q_j$, $S_j = \partial B_j$, $S = \bigcup_j S_j$. In changing G by a small isotopy we may assume that S is transversal to $f^{-k}(G)$ for all $k \in \mathbb{N}$. Then $S \cap (\bigcup_{k\in\mathbb{N}} f^{-k}(G))$ consists of a finite family $\mathcal{C}$ of closed curves. We have two cases (1) $\mathcal{C} = \emptyset$, (2) $\mathcal{C} \neq \emptyset$.

In case (1), N will denote the least integer such that $f^N(B) \subset int\ Q$. We have two subcases (1a) $N = 1$ and (1b) $N > 1$. We first consider (1a). It is known that the domain $B \setminus int\ Q$ is diffeomorphic to a union of k_ω copies of $S^2 \times [0,1]$ (see [10], chap. 8[13]). Hence there is a smooth function $\varphi : B \to \mathbb{R}$ extending $\varphi_{\mathcal{O}(\omega)}|_Q$ and having no critical point in $B \setminus int\ Q$. We claim that φ is a self-indexing energy function for $f|_B$. Indeed, for $x \in Q \setminus \mathcal{O}(\omega)$, $\varphi(f(x)) < \varphi(x)$ as it is true for $\varphi_{\mathcal{O}(\omega)}$. When $x \in B \setminus Q$, we have $\varphi(f(x)) < \varphi(\partial Q) < \varphi(x)$.

Let us consider case (1b) and set $\widetilde{B} = f^{N-1}(B)$. By the construction $\widetilde{B}$ is f-compressed and satisfies to condition (1a). Hence there is a self-indexing energy function $\tilde{\varphi} : \widetilde{B} \to \mathbb{R}$ for $f|_{\widetilde{B}}$. For any $x \in B$, we define $\varphi(x) = \tilde{\varphi}(f^{N-1}(x))$. It is easy to check that φ is the required function.

Let us consider case (2). A curve $C \in \mathcal{C}$ is said *innermost* on S if C bounds a disk $D_C \subset S$ whose interior contains no intersection curves from $\mathcal{C}$. Consider such an innermost curve. We have $C \subset f^{-k_C}(G)$ for some integer k_C and $f^{k_C}(C)$ is an innermost curve on $f^{k_C}(S)$. There is a unique disk E_C in G which is bounded by $f^{k_C}(C)$ such that the sphere $f^{k_C}(D_C) \cup E_C$ is homotopic to zero in $W^s(\mathcal{O}(\omega)) \setminus \mathcal{O}(\omega)$. We define $G' = (G \setminus E_C) \cup f^{k_C}(D_C)$. It bounds $Q' \subset W^s(\mathcal{O}(\omega))$, a union of 3-balls which contains $\mathcal{O}(\omega)$. The fact that C is an innermost curve implies $f(Q') \subset int\ Q'$.

There are two occurrences: (2a) $f(Q) \subset Q' \subset Q$ and (2b) $Q \subset Q' \subset f^{-1}(Q)$. In both cases there is a smooth approximation, $\widetilde{Q}$ of Q' such that $\widetilde{Q} \subset int\ Q'$ in case (2a) and $Q' \subset int\ \widetilde{Q}$ in case (2b); $\widetilde{Q}$ is still f-compressed. Set $\widetilde{G} = \partial\widetilde{Q}$. According to item (1a), $f^{-1}(\widetilde{G})$ in case (2a) and $\widetilde{G}$ in case (2b) is a level set of some self-indexing energy function defined respectively on $f^{-1}(\widetilde{Q})$ and on $\widetilde{Q}$. By the construction the number of curves in $S \cap (\bigcup_{k\in\mathbb{N}} f^{-k}(\tilde{G}))$ is less than in $\mathcal{C}$. We will repeat this process until getting a union of 3-balls $\widehat{Q}$ which is f-compressed and whose boundary $\widehat{G}$ does not intersect $f^k(S)$ for any k. Then we are reduced to case (1) and the lemma is proved. $\diamond$

4.2 A nice neighborhood of the attractor $\mathcal{A}(f)$ (or the repeller $\mathcal{R}(f)$)

Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism whose 1-dimensional separatrices are almost tame. Let us construct a "nice" neighborhood of the

[13]It is proved in this chapter that any smooth embedding of a ball into the interior of the standard ball is isotopic to a round ball — a result of J. W. Alexander, 1923. It is also proved that the isotopy extends as an ambient isotopy; hence the claim about the complement of a ball follows.

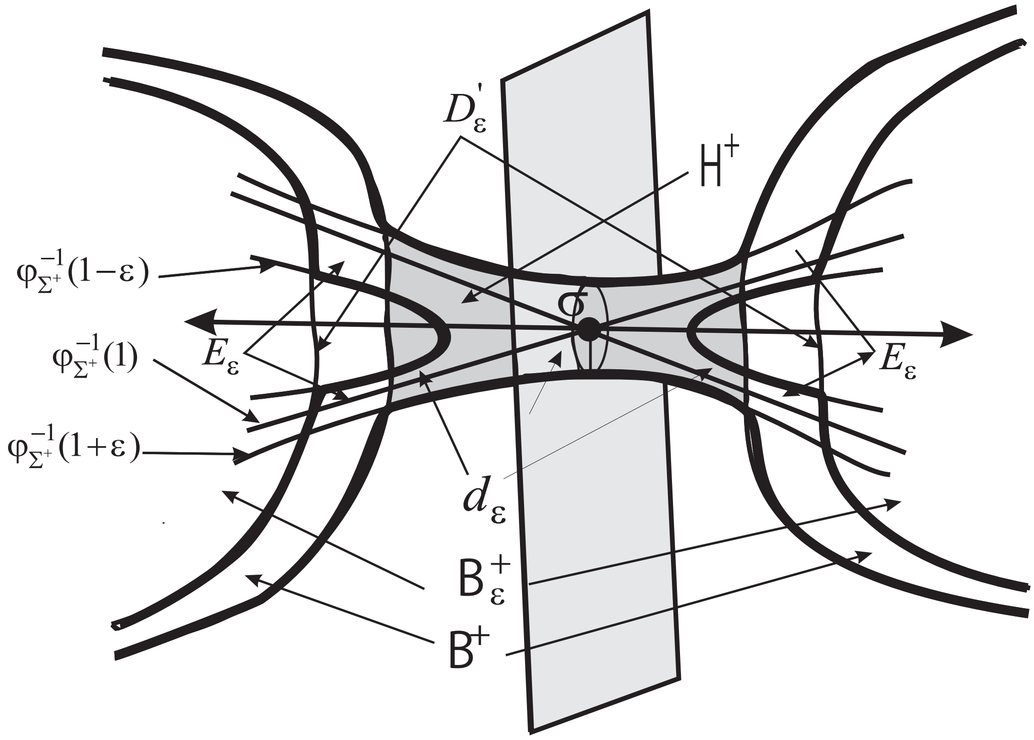

Figure 3: Construction of a nice neighborhood

attractor $\mathcal{A}(f)$.

According to lemma 2.2, each orbit $\mathcal{O}(\sigma), \sigma \in \Sigma^+$, has a neighborhood $U_{\mathcal{O}(\sigma)} \subset M^3$ endowed with a Morse-Lyapunov function $\varphi_{\mathcal{O}(\sigma)} : U_{\mathcal{O}(\sigma)} \to \mathbb{R}$ of f. Set $U_{\Sigma^+} = \bigcup\limits_{\sigma \in \Sigma^+} U_{\mathcal{O}(\sigma)}$ and denote $\varphi_{\Sigma^+} : U_{\Sigma^+} \to \mathbb{R}$ the function made of the union of the $\varphi_{\mathcal{O}(\sigma)}$'s.

Each connected component U_σ of U_{Σ^+}, $\sigma \in \Sigma^+$, is endowed with Morse coordinates (x_1, x_2, x_3) as in the conclusion of lemma 2.2: $\varphi_{\Sigma^+}(x_1, x_2, x_3) = 1 - x_1^2 + x_2^2 + x_3^2$, the x_1-axis is contained in the unstable manifold and the (x_2, x_3)-plane is contained in the stable manifold.

According to lemma 4.1, there exists an f-compressed domain B^+, made of $|\Omega^+|$ balls, which is a neighborhood of Ω^+ and such that each separatrix $\ell \in L^+$ intersects ∂B^+ in one point a_ℓ only. Due to the λ-lemma[14] (see, for example, [14]), replacing B^+ by $f^{-n}(B^+)$ for some $n > 0$ if necessary, we may assume that ∂B^+ are transverse to the regular part of the level set $C := \varphi^{-1}_{\Sigma^+}(1)$ and each of the intersections $C \cap \partial B^+$ consists of $2|\Sigma^+|$ circles. Due to lemma 4.2 there is a self-indexing energy function $\varphi_{B^+} : B^+ \to \mathbb{R}$ with a level set ∂B^+. For $\varepsilon > 0$ set $B^+_\varepsilon = \varphi^{-1}_{B^+}([0, \varphi_{B^+}(\partial B^+) - \varepsilon])$ and

[14]The λ-lemma claims that $f^{-n}(\partial B^+) \cap U_\sigma$ tends to $\{x_1 = 0\} \cap U_\sigma$ in the C^1 topology when n goes to $+\infty$.

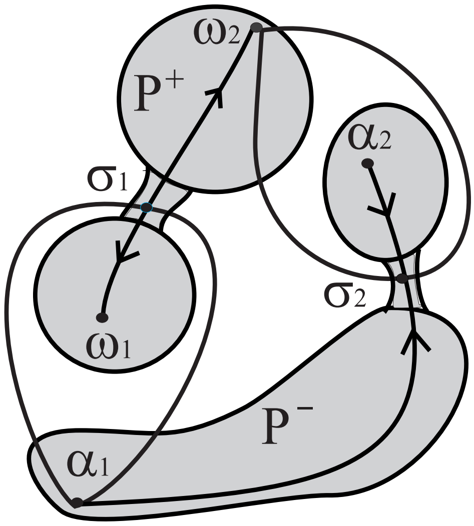

Figure 4: A pair of nice neighborhoods (P^+, P^-)

$E_\varepsilon = (B^+ \setminus int\ (B^+_\varepsilon)) \cap \{1-\varepsilon \leq \varphi_{\Sigma^+} \leq 1+\varepsilon\}$. We choose $\varepsilon > 0$ such that:

1) ∂B^+ and ∂B^+_ε are transverse to the level sets $\varphi^{-1}_{\Sigma^+}(1\pm\varepsilon)$, $f(B^+) \subset int\ B^+_\varepsilon$ and $(B^+ \setminus int\ (B^+_\varepsilon)) \setminus \{\varphi_{\Sigma^+} < 1-\varepsilon\}$ is a product;

2) $\varphi(f^{-1}(E_\varepsilon)) > 1+\varepsilon$ (it is possible as $\varphi(f^{-1}(C \setminus \Sigma^+)) > 1$).

We introduce H^+, the closure of $\{x \in U_{\Sigma^+} \mid x \notin B^+,\ \varphi_{\Sigma^+}(x) \leq 1+\varepsilon\}$ (see figure 3). By construction there is a smoothing P^+ of $B^+ \cup H^+$ such that:

- P^+ is f-compressed;

- P^+ is connected (see, for example, [3], lemma 1.3.3));

- P^+ is a handlebody of genus $|\Sigma^+| - |\Omega^+| + 1$[15], that is $g(f)$.

We call P^+ a *nice neighborhood* of the attractor $\mathcal{A}(f)$ (see figure 4). Making a similar construction for f^{-1} we obtain a nice neighborhood P^- of the repeller $\mathcal{R}(f)$, which is also a handlebody of genus $g(f)$ (lemma 3.3).

4.3 Construction of a self-indexing energy function on P^+ and P^-

Denote d_ε the part of the level set $\varphi^{-1}_{\Sigma^+}(1-\varepsilon)$ belonging to $U_{\Sigma^+} \setminus int\ B^+_\varepsilon)$. By construction d_ε is the union of $2|\Sigma^+|$ disks. Denote D'_ε the union of disks in ∂B^+_ε such that $\partial d_\varepsilon = \partial D'_\varepsilon$. We form S, a union of spheres, by removing the

[15]By the Mayer-Vietoris exact sequence $\beta_0 - \beta_1 = |\Omega^+| - |\Sigma^+|$, where β_0, β_1 are the Betti numbers of P^+. Take account of that $\beta_0 = 1$ and β_1 is the genus of P^+.

interiors of the D'_ε from ∂B^+_ε and gluing the d_ε. Denote $B(S)$ the union of balls bounded by S and containing Ω^+. We check that $B(S)$ is f-compressed. Indeed, it is true for B^+_ε. Moreover, $f(d_\varepsilon)$ does not intersect d_ε nor $\partial B^+_\varepsilon \setminus D'_\varepsilon$. The first intersection is empty as φ_{Σ^+} is a Lyapunov function and d_ε lies in a level set of it. The second one is empty as $\varphi_{\Sigma^+}(d_\varepsilon) = 1-\varepsilon \leq \varphi_{\Sigma^+}(x)$, for any $x \in U_{\Sigma^+}$, $x \in \partial B^+_\varepsilon \setminus D'_\varepsilon$.

Let K be the domain between ∂P^+ and S. We define a function $\varphi^+ : K \to \mathbb{R}$ whose value is $1+\varepsilon$ on ∂P^+, $1-\varepsilon$ on S, coinciding with φ_{Σ^+} on $K \cap H^+$ and without critical points outside H^+. This last condition is easy to satisfy as the domain in question is a product cobordism. With all the informations that we have on the image of f, it is easy to check that φ^+ is a Lyapunov function. Indeed, it is obvious for the points of K which are not in H^+. Suppose that $x \in K \cap H^+$. Then we have two possibilities: a) $f(x) \in H^+$; b) $f(x) \notin H^+$. In the first case the conclusion follows from the Lyapunov property of φ_{Σ^+}. In the second case we are going to show that $f(x) \in \{\varphi_{\Sigma^+} < 1-\varepsilon\}$ and then the conclusion also holds. Suppose on the contrary that $f(x) \notin \{\varphi_{\Sigma^+} < 1-\varepsilon\}$. Then $f(x)$ belongs to the domain E_ε. But it follows from the choice of E_ε that $f^{-1}(E_\varepsilon)$ does not intersect H^+. We get a contradiction.

According to lemma 4.2, there is an extension of φ^+ to $B(S)$, which is a self-indexing Morse-Lyapunov function. Finally, we get the desired self-indexing energy function on P^+.

As the set $\mathcal{R}(f)$ is an attractor of f^{-1} and P^- is a nice neighborhood of $\mathcal{R}(f)$, it is possible to construct (as above) a self-indexing energy function $\hat{\varphi}^- : P^- \to \mathbb{R}$ of f^{-1}. It follows from lemma 2.1 that the function $\varphi^- : P^- \to \mathbb{R}$ given by the formula $\varphi^-(x) = -\hat{\varphi}^-(x) + 3$ is a self-indexing energy function of f on P^-.

4.4 Proof of theorem 3

The main assumption, which is not a necessary condition, is the following: $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is diffeomorphic to the product $S_{g(f)} \times \mathbb{R}$.

Let us denote $S^\pm = \partial P^\pm$. It is easy to arrange that $\varphi^-(S^-) > \varphi^+(S^+)$.

First assume $(*)$ $\quad S^- \cap \big(\bigcup\limits_{k>0} f^{-k}(S^+)\big) = \emptyset$. Let m be the first positive integer such that $f^m(S^-) \subset \mathit{int}\ P^+$. If $m = 1$, as N is a product, there exists a smooth function $\varphi : N \to \mathbb{R}$ without critical points which extends $\varphi^+ \cup \varphi^- : P^+ \cup P^- \to \mathbb{R}$. It is a Lyapunov function as $f(N) \subset P^+$ and $f^{-1}(N) \subset P^-$.

If $m > 1$, the surfaces $f^{-1}(S^+), f^{-2}(S^+), \dots, f^{-m+1}(S^+)$ are mutually "parallel", that is: two by two they bound a product cobordism, diffeomorphic to $S_{g(f)} \times [0,1]$ (see for instance theorem 3.3 in [8]). Therefore they subdivide

N in product cobordisms and there exists a function φ extending $\varphi^{\pm}$ on $P^{\pm}$, without critical points on N and having $f^{-1}(S^+), \ldots, f^{-m+1}(S^+)$ as level sets. One easily checks that such a φ is a self-indexing energy function for f.

We now explain the end of the proof, that is, how to reduce oneself to $(*)$. Without loss of generality we may assume that S^- is transversal to $\bigcup_{k>0} f^{-k}(S^+)$, which implies that there is a finite family $\mathcal{C}$ of intersection curves. We are going to describe (as in lemma 4.2) a process decreasing the number of intersection curves by an isotopy of P^+ among handlebodies which are f-compressed; they will be all equipped with a self-indexing energy function which is constant on the boundary.

For simplifying the statement of the next lemma, we use the following definition.

Definition 4.3 *Let S be a proper bicollared embedded surface in a 3-manifold W (proper meaning $\partial S \subset \partial W$ when $\partial S \neq \emptyset$). One says that S is incompressible in W if any simple curve γ in S, which bounds an embedded disk in W starting on one side of S along its boundary, is homotopic to zero in S, and hence bounds an embedded disk in S*[16].

For instance, our S^+ and S^- are incompressible in N.

Lemma 4.4
1) Any 2-sphere which is embedded in N, P^+ or P^- bounds a ball there.
2) S^+ (resp. S^-) is incompressible in $P^+ \setminus \mathcal{A}(f)$ (resp. $P^- \setminus \mathcal{R}(f)$).
3) S^+ is incompressible in $N \cup (P^- \setminus \mathcal{R}(f))$.
4) Both S^+ and S^- and their images by $f^k, k \in \mathbb{Z}$, are incompressible in $M \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$.

Proof:
1) As each of the considered domains embeds into $\mathbb{R}^3$, every embedded sphere bounds a 3-ball (generalized Schönflies theorem[17][4]).

2) Let γ be a simple curve in S^+ which bounds a disk δ in $P^+ \setminus \mathcal{A}(f)$. In changing δ by a small isotopy we may assume that δ is transverse to $W^s(\Sigma^+)$; so $\delta \cap W^s(\Sigma^+)$ consists of a finite family $\mathcal{I}$ of arcs with end points in γ and a finite family $\mathcal{L}$ of closed curves. Each arc $\alpha \in \mathcal{I}$, after some isotopy in $W^s(\Sigma^+)$ pushing α into S^+, indicates a way of decomposing γ as a connected

[16]It is well known by topologists that this definition is equivalent to the fact that the inclusion $S \hookrightarrow W$ induces an injection of fundamental groups; but we do not use this deep result.

[17] In [4] M. Brown proved the topological statement. A smooth version of this result is available in [5].

sum $\gamma_1\#\gamma_2$ of two simple curves of S^+ bounding disks in V. Of course, if the conclusion of 2) in lemma 4.4 holds for both curves γ_1 and γ_2, it also does for γ. Finally, one reduces oneself to consider the case when $\gamma \cap W^s(\Sigma^+) = \emptyset$ (that is, $\mathcal{I} = \emptyset$); thus, γ can be thought of as a curve in ∂B^+ where B^+, a union of 3-balls, is obtained from P^+ by cutting along the disks $P^+ \cap W^s(\Sigma^+)$. Similarly, by cut-and-paste, it is possible to remove from $\mathcal{L}$ the closed curves which bound disks in $W^s(\Sigma^+) \setminus \Sigma^+$.

First, assume that $\mathcal{L}$ is empty; in other words, $\delta \subset B^+$. Thus, there are two disks d' and d'' in ∂B^+ which is bounded by γ. According to item 1), δ divides one component of B^+ into two balls B' and B'', with $d' \subset B'$ and $d'' \subset B''$. Since $\delta \cap \mathcal{A}(f)$ is empty and each component of B^+ contains exactly one connected component of $B^+ \cap \mathcal{A}(f)$, one of B' or B'' is disjoint from $\mathcal{A}(f)$. If it is B', that means that d' is a disc in the boundary of P^+. Moreover B' is a ball in $P^+ \setminus \mathcal{A}(f)$.

In order to finish the proof, we have to consider the case when $\mathcal{L}$ is not empty, but made of curves which bound disks in $W^s(\Sigma^+)$ each one having one saddle point in its interior. Let c be such a curve which is innermost in δ; it bounds a disk $d_c \subset W^s(\Sigma^+)$ and a disk $\delta_c \subset \delta$. Let $\sigma \in \Sigma^+$ be the saddle point in d_c and set $D(\sigma) = P^+ \cap W^s(\sigma)$. After smoothing and a small isotopy, $d_c \cup \delta_c$ gives rise to an embedded 2-sphere $\mathcal{S}$ which is disjoint from $W^s(\Sigma^+)$ and, hence, lies in a connected component B_0 of B^+. As $\mathcal{S}$ intersects $\mathcal{A}(f)$ in one point exactly, both separatrices of σ must enter two different connected components of B^+, one being B_0 and the other being denoted B_1. Then $D(\sigma)$ decomposes P^+ as a connected sum $P^+ = P_0\#P_1$, with $P_j \supset B_j$ for $j = 0, 1$.

The sphere $\mathcal{S}$ bounds a 3-ball $\mathcal{B}$ in B_0, but, since there is some separatrix of σ which enters $\mathcal{B}$ without getting out, $\mathcal{B}$ must contain one sink ω_0; moreover, since δ_c avoids $\mathcal{A}(f)$, ω_0 is in the closure of no other separatrix. Hence, P_0 is a ball. If $\gamma \subset \partial P_0$, there is nothing to do; so, assume $\gamma \subset \partial P_1$. Since ∂P_0 is a 2-sphere, it is equivalent that γ bounds a disk in ∂P^+ or in ∂P_1. This allows us to ignore $W^s(\sigma) \cup W^s(\omega_0)$. Repeating this process we are reduced to the case when $\mathcal{L}$ is empty.

3) As N is a product, S^+ and S^- are clearly incompressible in N. One looks at δ, a disk in $N \cup P^-$ whose boundary lies in S^+, and at its intersection curves with S^-. Using 2) and the innermost curve techniques, one reduces to the case $\delta \subset N$ and the conclusion follows.

4) For proving the statement for S^+, we take δ, an embedded disc in $M \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ with boundary in S^+ and a collar of $\partial\delta$ transverse to S^+. Then, in general position, we have finitely many intersection curves in $int\,\delta \cap (S^+ \cup S^-)$. By using 2) and 3) one eliminates successively all intersection curves. Finally, δ lies in N or $P^+ \setminus \mathcal{A}(f)$ and the conclusion follows. $\diamond$

Lemma 4.4 allows us to remove all intersection curves which are homotopic to zero in S^+ or, equivalently, in $f^k(S^-)$, $k > 0$ (as in the proof of lemma 4.2). We recall m, the largest integer such that $f^m(S^-) \cap S^+ \neq \emptyset$. Let F^- be a connected component of $f^m(S^-) \cap N$. Since $f^m(S^-) \cap S^- = \emptyset$, we have $\partial F^- \subset S^+$. We claim that F^- in incompressible in N. Indeed, if δ is a disk in N with boundary $\gamma \subset F^-$, according to 3) in lemma 4.4, γ is homotopic to zero in $f^m(S^-)$. As none of the components of ∂F^- is homotopic to zero, γ is homotopic to zero in F^-.

Therefore, according to F. Waldhausen (corollary 3.2 in [20]), there is some surface $F^+ \subset S^+$ diffeomorphic to F^-, with $\partial F^+ = \partial F^-$, and $F^+ \cup F^-$ bounds a domain Δ in N, which, up to smoothing of the boundary, is diffeomorphic to $F^- \times [0,1]$. We then change S^+ to S' by removing the interior of F^+ and gluing F^-. After a convenient smoothing, this surface S' has less intersection curves with $\bigcap\limits_k f^k(S^-)$ than S^+. By construction, it bounds a handlebody P' which is isotopic to P^+ and f-compressed; moreover P' carries a self-indexing Lyapunov function. Arguing recursively, we are reduced to case $(*)$. In this final recursive argument, once lemma 4.4 is proved, it is no longer usefull that P^+ intersects $W^s(\Sigma^+)$ along disks. This finishes the proof of theorem 3.

4.5 Proof of theorem 4

The necessary condition of theorem 4 is yielded by lemma 3.4. For proving that it is sufficient, we have to construct a self-indexing energy function on M^3 under the assumptions of theorem 4 that we recall now. The diffeomorphism $f : M^3 \to M^3$ is a gradient-like diffeomorphism and M^3 is the union of three domains with mututally disjoint interiors, $M^3 = P^+ \cup N \cup P^-$, satisfying the following conditions.

1) P^+ (resp. P^-) is a f-compressed (resp. f^{-1}-compressed) handlebody of genus $g(f)$ and $\mathcal{A}(f) \subset P^+$ (resp. $\mathcal{R}(f) \subset P^-$);

2) $W^s(\sigma^+) \cap P^+$ (resp. $W^u(\sigma^-) \cap P^-$) consists of exactly one two-dimensional closed disk for each saddle point $\sigma^+ \in \Sigma^+$ (resp. $\sigma^- \in \Sigma^-$);

3) there is a diffeomorphism $q : S_{g(f)} \times [0,1] \to N$ such that $q(S_{g(f)} \times \{t\})$, $t \in [0,1]$ bounds an f-compressed handlebody.

Our assumption makes P^+ (resp. P^-) very close to a nice neighborhood of $\mathcal{A}(f)$ (resp. $\mathcal{R}(f)$) in the sense of 4.2. If we remove from P^+ a thin neighborhood of the f-compressed union of disks $P^+ \cap W^s(\Sigma^+)$, we get a f-compressed domain B^+, union of $|\Omega^+|$ balls, such that ∂B^+ intersects each separatrix $\ell \in L^+$ in one point only. Adding H^+ to it (as in 4.2), we get a new handlebody, we still denote P^+, which is a genuine nice neighborhood of $\mathcal{A}(f)$. We perform a similar change on P^- and the complement remains a product. We can construct self-indexing energy functions $\varphi^+ : P^+ \to \mathbb{R}$ and $\varphi^- : P^- \to \mathbb{R}$

as in 4.3 which are constant on their respective boundaries. Let us denote $S^{\pm} = \partial P^{\pm}$. It is easy to arrange that $\varphi^-(S^-) > \varphi^+(S^+)$. Finally we can extend $\varphi^{\pm}$ to N due to condition 3) of the theorem.

5 Example

In this section we construct an example of a Morse-Smale diffeomorphism f on $M^3 = \mathbb{S}^2 \times \mathbb{S}^1$ possessing an energy function and such that $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is not a product. More precisely we prove next proposition.

Proposition 5.1 *There exists a Morse-Smale diffeomorphism $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ with the following properties:*

1) the non-wandering set $\Omega(f)$ consists of four hyperbolic fixed points: ω is a sink, σ^+ and σ^- are saddles of respective indices 1 and 2, α is a source, hence f is gradient-like diffeomorphism for which $\mathcal{A}(f) = W^u(\sigma^+) \cup \{\omega\}$, $\mathcal{R}(f) = W^s(\sigma^-) \cup \{\alpha\}$ and $g(f) = 1$ (with notation introduced after theorem 2);

2) $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is not diffeomorphic to the product $\mathbb{T}^2 \times \mathbb{R}$ (here $\mathbb{T}^2$ is the two-dimensional torus) and f satisfies the conditions of theorem 4 (hence it possesses an energy function).

Proof: We first define f^+ on a 3-ball B^+ as the homothety centered at ω of ratio 1/2. Let A^+ be the closure of $B^+ \setminus f^+(B^+)$; it is a fundamental domain for $f^+|_{B^+}$. Let d_1^+, d_2^+ be two disjoint disks in ∂B^+, with respective centers a_1^+, a_2^+, which are used as attaching disks for a 1-handle $H^+ \cong [-1, +1] \times D^2$, where D^2 is a 2-disk. We have to extend f^+ to $P^+ = B^+ \cup H^+$ so that the point $\{0\} \times \{0\}$ of H^+ is a hyperbolic fixed point of index 1, with the core of H^+ as the local unstable manifold and the meridian disk $\Delta^+ = \{0\} \times D^2$ of H^+ as the stable manifold (take for instance $f^+|_{\Delta^+}$ as being the 1/2-contraction). This extension will be essentially determined once we define the embedding $f^+ : ([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]) \times D^2 \to A^+$, that is: a pair of disjoint tubes in A^+ joining $\frac{1}{2}d_1^+$, $\frac{1}{2}d_2^+$ to $f^+(d_1^+)$, $f^+(d_2^+)$ respectively. We describe below the cores of these tubes.

For them, we choose a so-called *string link* C^+ formed with a pair of disjoint arcs (c_1^+, c_2^+) in A^+, each one joining a_i^+, $i = 1, 2$ to its image $f^+(a_i^+)$. The following properties are required:

i) the pair (A^+, C^+) is not a product $(\mathbb{S}^2 \setminus \{x, y\}) \times [0, 1]$;

ii) there exists an involution $I^+ : (A^+, C^+) \to (A^+, C^+)$ permuting both boundary components of A such that $I^+|_{\partial B^+} = f^+$.

An example of such a string link there is shown on figure 5 for which involution I^+ is mirror image with respect to middle sphere of A^+. By construction, f^+ is a compression of P^+ with two hyperbolic fixed points, a sink ω and a saddle σ^+ of index 1. The unstable manifold of σ^+ consists of the core of H^+

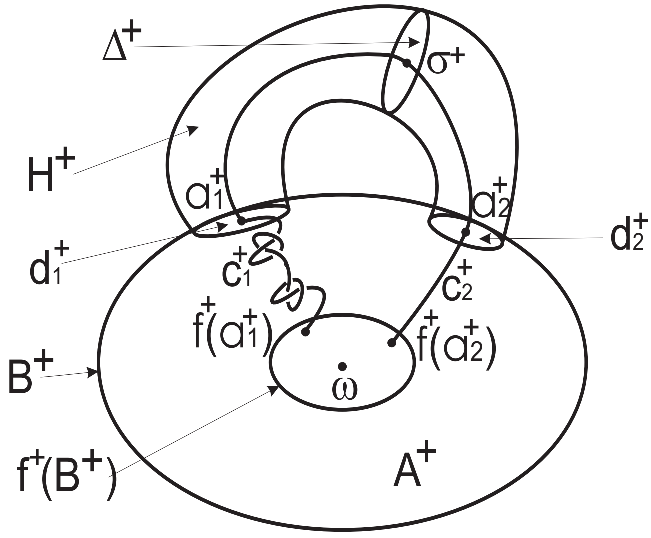

Figure 5: String link

and the union $\bigcup_{n\in\mathbb{N}}(f^+)^n(C^+)$. Let W^+ be the closure of $P^+ \setminus f^+(P^+)$; it is bounded by two tori.

Lemma 5.2
1) The domain W^+ is not a product $\mathbb{T}^2 \times [0,1]$.
2) There is an involution $J^+ : W^+ \to W^+$ which permutes both boundary components such that $J^+|_{\partial P^+} = f^+$.

Proof:
1) We can see $f^+(P^+)$ as the tubular neighborhood of a closed curve κ^+ in P^+ which intersects Δ^+, the meridian disk of P^+, in one point only, namely σ^+. By cutting P^+ along Δ^+, we get a 3-ball $Q^+ \cong B^+$ and a relative knot $\kappa'^+ = \kappa^+ \cap Q^+$ which consists of the union of c_1^+, c_2^+ and an unknotted arc in $f^+(\partial B^+)$ joining $f^+(a_1^+)$ to $f^+(a_2^+)$. If we cut $f^+(P^+)$ along Δ^+, we get a tubular neighborhood of κ'^+. It is easy to prove that condition i) on the chosen string link C^+ is equivalent to the following i'):

i') there is no embedded disk in Q^+ whose boundary consists of κ'^+ and one arc in ∂Q^+.

Assume that W^+ is a product. Then there exists a 2-annulus R^+ with one boundary component in ∂P^+ and the other consisting of κ^+. By usual techniques, the intersection $R^+ \cap \Delta^+$ can be reduced to an arc joining both boundary components of R^+. Thus, cutting R^+ along Δ^+ yields a disk in Q^+ whose existence is forbidden by i').

2) Let N^+ be a tubular neighborhood of C^+ in A^+ which is invariant by I^+. The end fibers of N^+ consist of the disks d_1^+, d_2^+ and their images by f^+.

Another description of W^+ is the following. We remove the interior of N^+ (that is, the open tubular neighborhood) and, along $\partial N^+ \cong \mathbb{S}^1 \times [0,1] \times \{-1,1\}$, we glue $H'^+ := \mathbb{S}^1 \times [0,1] \times [-1,1]$. In this description of W^+, the restriction of f^+ to $\partial P^+ \cap H^+$ is the "identity" of $\mathbb{S}^1 \times \{0\} \times [-1,1] \to \mathbb{S}^1 \times \{1\} \times [-1,1]$. On the other hand, the involution I^+, restricted to ∂N^+ is conjugate to $Id|_{\mathbb{S}^1} \times \tau$, where τ is the standard involution of the interval $[0,1]$. Therefore, I^+ extends to H'^+ as an involution J^+ which is the "identity" from $\partial P^+ \cap H'^+ \cong \mathbb{S}^1 \times \{0\} \times [-1,1]$ to $f^+(\partial P^+) \cap H'^+ \cong \mathbb{S}^1 \times \{1\} \times [-1,1]$. Finally $J^+ = f^+$ on ∂P^+. $\diamond$

Now, let us consider the quotient W^+/f^+ and the natural projection $p^+ : W^+ \to W^+/f^+$. By the above construction $T^+ = p^+(\Delta^+ \cap W^+)$ is a 2-torus. Let $V^+ \cong T^+ \times [-1,1]$ be a tubular neighborhood of T^+ in W^+/f^+ and $\hat{h} : W^+/f^+ \to W^+/f^+$ be a diffeomorphism such that $\hat{h}$ preserves p^+, $\hat{h} = id$ outside of $int\ V^+$ and $\hat{h}(T^+) \cap T^+ = \emptyset$. Then the lift $h : W^+ \to W^+$ of $\hat{h}$ preserves ∂P^+, commutes with f^+ and $h(\Delta^+ \cap W^+) \cap \Delta^+ = \emptyset$.

We now finish the construction of our example. We consider a new copy P^- of P^+. We glue them by $h \circ J^+$, viewed as a diffeomorphism $W^- \to W^+$ where W^- is the copy of W^+ in P^-. Let $f^- : P^- \to P^-$ be the copy of f^+. Hence, our ambient manifold is $M^3 = P^- \underset{h \circ J^+}{\cup} P^+$ and the diffeomorphism $f : M^3 \to M^3$ is defined by $f|_{P^+} = f^+$ and $f|_{P^-} = (f^-)^{-1}$; one easily checks that both definitions fit together. Our f is a Morse-Smale diffeomorphism as the unstable manifold of σ^- avoids the stable manifold of σ^+.

The repeller $\mathcal{R}(f)$ is the attractor of f^-, that is the copy of $\mathcal{A}(f)$ in P^-. By changing P^- into $f^{-1}(P^-)$, then P^+ and $f^{-1}(P^-)$ are no longer overlaping; they only have a common boundary and the assumptions of theorem 4 are satisfied. This example is the desired one as $M^3 \setminus (\mathcal{A}(f) \cup \mathcal{R}(f))$ is not the product $\mathbb{T}^2 \times \mathbb{R}$. Indeed, if it would be a product, then W^+ itself should be one, contradicting the preceding lemma. $\diamond$

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